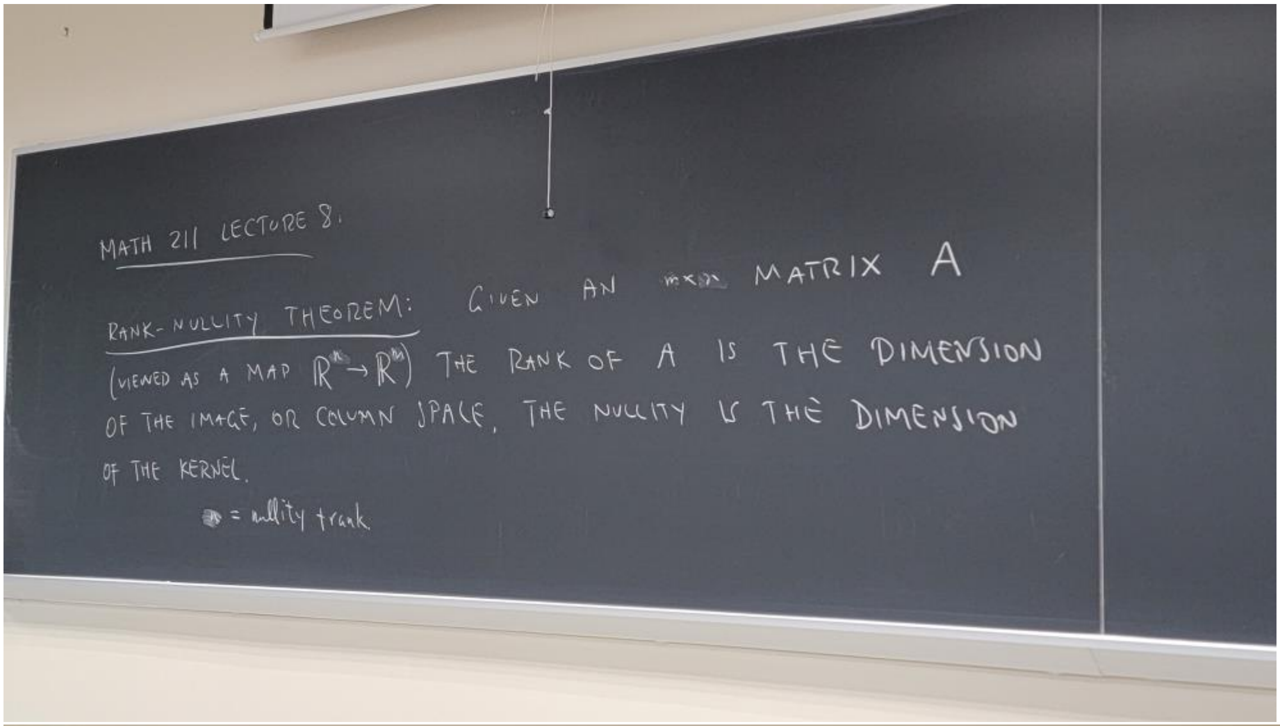
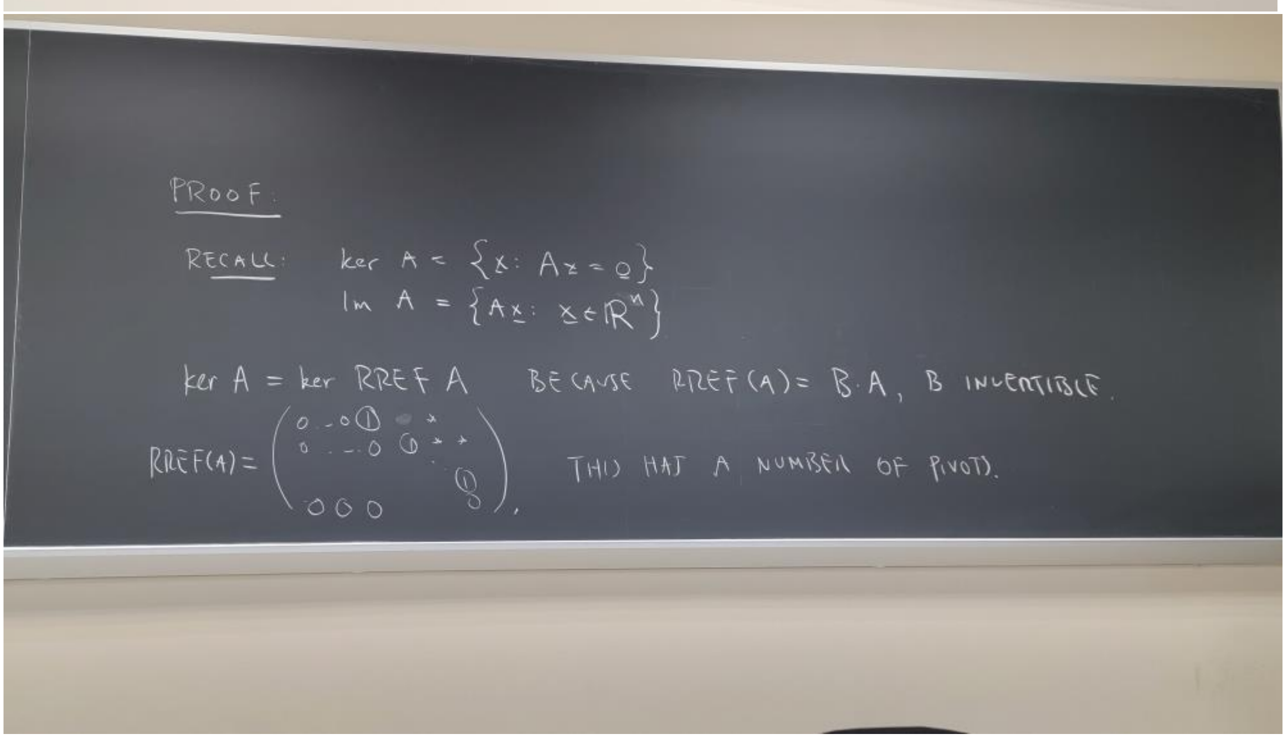
Lecture #8 (2/16/23)

Friday, February 17, 2023 8:35 PM





SOLVING $RREF(A)\cdot \underline{x} = \underline{0}$, YOU CAN SOLVE FOR EACH PIVOT VARIABLE IN TERMS OF THE OTHER VARIABLES.

SO $\text{nullity}(A) = n - \#\text{PIVOTS} = \#$ FREE VARIABLES.

THOSE COLUMNS OF A WHICH SPAN $\text{Im}(A)$ CAN BE CHOSEN TO BE THOSE CONTAINING PIVOTS. THIS IS OBVIOUS FOR RREF(A) WHERE THE PIVOT COLUMNS ARE LINEARLY INDEPENDENT, BECAUSE THE PIVOT COLUMNS HAVE A 1 WHERE PREVIOUS COLUMNS ARE 0.

ALSO, CAN USE THE PIVOT COLUMNS TO SPAN BY DOING COLUMN REDUCTION. THE IMAGE SPACE HAS AT MOST DIMENSION #PIVOTS BECAUSE THE ROWS HAVE ONLY THIS MANY NON-ZERO ENTRIES.

$$\begin{pmatrix} 0 \cdots 0 \; 1 & * & * \\ 0 \text{———} & 1 & * \\ \text{————} & & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} d \text{ PIVOTS.} \\ \subset \mathbb{R}^d \times \{\underline{0}\}. \end{matrix}$$

A LINEAR RELATION AMONG THE COLUMNS OF A IS EXPRESSED $A\underline{x} = \underline{0}$.

THIS HAS THE SAME LINEAR RELATION FOR $RREF(A) = B\cdot A$.

$BA\underline{x} = 0$. THUS IF SOME COLUMNS OF RREF(A) ARE A BASIS THE SAME COLUMNS ARE A BASIS FOR A.

$$\underbrace{n = \text{nullity} + \text{rank}}_{}$$

A LINEAR RELATION AMONG THE COLUMNS OF A

IS EXPRESSED $A\underline{x} = \underline{0}$.

THIS HAS THE SAME LINEAR RELATION FOR $RREF(A) = B \cdot A$.

$BA\underline{x} = 0$. THUS IF SOME COLUMNS OF $RREF(A)$ ARE A BASIS

THE SAME COLUMNS ARE A BASIS FOR A.

$$n = \underset{n - \#\text{PIVOTS}}{\overset{\#\text{FREE VARIABLES}}{\text{nullity}}} + \underset{\#\text{PIVOTS}}{\text{rank}}$$

$\square$

THEOREM: THE VECTORS $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n \in \mathbb{R}^n$ FORM A BASIS IF AND ONLY IF

$$A = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix} \text{ IS INVERTIBLE.}$$

PROOF: n VECTORS IN $\mathbb{R}^n$ ARE A BASIS

$\Leftrightarrow$ THE VECTORS SPAN

$\Leftrightarrow$ THE VECTORS ARE LINEARLY INDEPENDENT.

THE LINEAR RELATION $x_1\underline{v}_1 + x_2\underline{v}_2 + \ldots + x_n\underline{v}_n = \underline{0}$

CAN BE EXPRESSED $A\underline{x} = \underline{0}$.

THE VECTORS ARE LINEARLY INDEP. $\Leftrightarrow \{x : Ax = 0\} = \{0\} \Leftrightarrow x \mapsto Ax$ IS 1-1 ONTO

PROOF: n VECTORS IN $\mathbb{R}^n$ ARE A BASIS

$\iff$ THE VECTORS SPAN

$\iff$ THE VECTORS ARE LINEARLY INDEPENDENT.

THE LINEAR RELATION $x_1\underline{v}_1 + x_2\underline{v}_2 + \dots + x_n\underline{v}_n = \underline{0}$

CAN BE EXPRESSED $A\underline{x} = \underline{0}$.

THE VECTORS ARE LINEARLY INDEP. $\iff \{\underline{x} : A\underline{x} = \underline{0}\} = \{\underline{0}\} \iff \underline{x} \mapsto A\underline{x}$ IS 1-1, ONTO

$\iff$ A IS INVERTIBLE. $\square$

DEFINITION: GIVEN k LINEARLY INDEPENDENT VECTORS $\underline{v}_1, \dots, \underline{v}_k$, THESE ARE A BASIS OF THE SUBSPACE $V = \mathrm{SPAN}\{\underline{v}_1, \dots, \underline{v}_k\}$ OF $\mathbb{R}^n$.

ANY $\underline{x} \in V$ MAY BE EXPRESSED UNIQUELY AS

$\underline{x} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots + c_k\underline{v}_k$ FOR SOME $c_1, \dots, c_k \in \mathbb{R}$.

THE VECTOR $\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ IS THE <u>COORDINATE VECTOR</u> OF $\underline{x}$ IN THE BASIS $\mathcal{B}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k)$.

EXAMPLE:

$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ SPANS $\mathbb{R}^3$.

$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ SPAN $\mathbb{R}^3$

$$\underline{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underline{(-1)} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \underline{3} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

EXAMPLE:

$$\mathcal{B}_1 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \quad \text{SPANS } \mathbb{R}^3.$$

$$\mathcal{B}_2 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \quad \text{SPAN } \mathbb{R}^3$$

$$\underline{-1} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + \underline{(-1)} \begin{pmatrix}1\\1\\0\end{pmatrix} + \underline{3} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

THE VECTOR $\begin{pmatrix}1\\2\\3\end{pmatrix}$ IN $\mathbb{R}^3$ HAS COOR. VECTOR $\begin{pmatrix}1\\2\\3\end{pmatrix}$ W.R.T. $\mathcal{B}_1$, IN $\mathcal{B}_2$, $\underline{c} = \begin{pmatrix}-1\\-1\\3\end{pmatrix}$.

GIVEN $\underline{v}_1, \ldots, \underline{v}_k$ A BASIS FOR V

AND $\underline{x} \in V$

$\underline{c}$ IS THE SOLUTION OF THE EQUATION

$$\begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_k \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \underline{x}.$$

WE WRITE $[\underline{x}]_{\mathcal{B}}$ FOR THE COORDINATE VECTOR WITH RESPECT TO $\mathcal{B}$.

THEOREM: COORDINATES ARE LINEAR.

GIVEN A BASIS $\mathcal{B} = \{\underline{v}_1, \ldots, \underline{v}_k\}$ FOR V

AND $\underline{x}, \underline{y} \in V$,

$$[\underline{x}]_{\mathcal{B}} + [\underline{y}]_{\mathcal{B}} = [\underline{x}+\underline{y}]_{\mathcal{B}}.$$

$$[a \cdot \underline{x}]_{\mathcal{B}} = a[\underline{x}]_{\mathcal{B}}, \quad a \in \mathbb{R}$$

THEOREM: COORDINATES ARE LINEAR.

GIVEN A BASIS $\mathcal{B}=\{\underline{v}_1,\dots,\underline{v}_k\}$ FOR V

AND $\underline{x},\underline{y}\in V$,

$$[\underline{x}]_{\mathcal{B}}+[\underline{y}]_{\mathcal{B}}=[\underline{x}+\underline{y}]_{\mathcal{B}}.$$

$$[a\cdot\underline{x}]_{\mathcal{B}}=a[\underline{x}]_{\mathcal{B}},\quad a\in\mathbb{R}$$

PROOF: $[\underline{x}]_{\mathcal{B}}=\begin{bmatrix}c_1\\ \vdots\\ c_k\end{bmatrix}$ SO THAT

$c_1\underline{v}_1+\dots+c_k\underline{v}_k=\underline{x}$.

$[\underline{y}]_{\mathcal{B}}=\begin{bmatrix}d_1\\ \vdots\\ d_k\end{bmatrix}$, $d_1\underline{v}_1+\dots+d_k\underline{v}_k=\underline{y}$.

THEN $\begin{pmatrix}c_1+d_1\\ c_2+d_2\\ \vdots\\ c_k+d_k\end{pmatrix}$ SATISFIES $(c_1+d_1)\underline{v}_1+\dots+(c_k+d_k)\underline{v}_k=\underline{x}+\underline{y}$.

$a\cdot\begin{pmatrix}c_1\\ \vdots\\ c_k\end{pmatrix}=\begin{pmatrix}ac_1\\ ac_2\\ \vdots\\ ac_k\end{pmatrix}$ SATISFIES $ac_1\underline{v}_1+\dots+ac_k\underline{v}_k=a\cdot\underline{x}$

EXAMPLE:

$\underline{v}_1=\begin{bmatrix}3\\1\end{bmatrix}$, $\underline{v}_2=\begin{bmatrix}-1\\3\end{bmatrix}$ THESE SPAN $\mathbb{R}^2$.

$$4\cdot\begin{bmatrix}3\\1\end{bmatrix}+2\begin{bmatrix}-1\\3\end{bmatrix}=\begin{pmatrix}12-2\\4+6\end{pmatrix}=\begin{bmatrix}10\\10\end{bmatrix}.\quad\checkmark$$

$\underline{x}=\begin{bmatrix}10\\10\end{bmatrix}$, $\begin{bmatrix}3&-1\\1&3\end{bmatrix}[\underline{x}]_{\mathcal{B}}=\begin{bmatrix}10\\10\end{bmatrix}$.

SWAP ROW 1,2 $\begin{pmatrix}1&3\\3&-1\end{pmatrix}\underline{c}=\begin{pmatrix}10\\10\end{pmatrix}\to\begin{bmatrix}1&3\\0&-10\end{bmatrix}\underline{c}=\begin{bmatrix}10\\-20\end{bmatrix}$ $c_2=2$.

EXAMPLE:

$\underline{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ THESE SPAN $\mathbb{R}^2$.

$\underline{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$. $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} [x]_{\mathcal{B}} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$.

SWAP ROW 1,2 $\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \underline{c} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -10 \end{bmatrix} \underset{c_1, c_2}{\underline{c}} = \begin{pmatrix} 10 \\ -20 \end{pmatrix}$

$c_2 = 2$.

$c_1 + 3c_2 = 10$, $c_1 = 4$.

$4 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{pmatrix} 12 - 2 \\ 4 + 6 \end{pmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$. ✓

THEOREM. LET $T: \mathbb{R}^n \to \mathbb{R}^n$.

$\mathcal{B}$ IS A BASIS $\underline{v}_1, \dots, \underline{v}_n$.

$$[T\underline{x}]_{\mathcal{B}} = B \cdot [\underline{x}]_{\mathcal{B}}.$$

WHERE B IS THE MATRIX $B = \begin{bmatrix} | & | & & | \\ [T\underline{v}_1]_{\mathcal{B}} & [T\underline{v}_2]_{\mathcal{B}} & \cdots & [T\underline{v}_n]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$

IF $\mathcal{B}$ IS THE STANDARD BASIS THIS THEOREM RECOVERS $\mathrm{MAT}(T) = \begin{bmatrix} | & & | \\ T\underline{e}_1 & \cdots & T\underline{e}_n \\ | & & | \end{bmatrix}$.

PROOF. LET $[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ SO $\underline{x} = c_1 \underline{v}_1 + \dots + c_n \underline{v}_n$.

$$T\underline{x} = c_1 T\underline{v}_1 + c_2 T\underline{v}_2 + \dots + c_n T\underline{v}_n$$

$$= \begin{bmatrix} T\underline{v}_1 & \cdots & T\underline{v}_n \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

TO OBTAIN COORDINATES WITH RESPECT TO $\underline{v}_1, \dots, \underline{v}_n$ WRITE $T\underline{v}_i = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix} [T\underline{v}_i]$

TO OBTAIN COORDINATES WITH RESPECT TO $\underline{v}_1, \ldots, \underline{v}_n$ WRITE $T\underline{v}_i = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix} [T\underline{v}_i]_{\mathcal{B}}$.

$$T\underline{x} = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix} \cdot \begin{bmatrix} | & & | \\ [T\underline{v}_1]_{\mathcal{B}} & \cdots & [T\underline{v}_n]_{\mathcal{B}} \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{[\underline{x}]_{\mathcal{B}}}.$$

THIS MEANS $(T\underline{x})_{\mathcal{B}} = \begin{bmatrix} | & & | \\ [T\underline{v}_1]_{\mathcal{B}} & \cdots & [T\underline{v}_n]_{\mathcal{B}} \\ | & & | \end{bmatrix} [\underline{x}]_{\mathcal{B}}$. $\square$

THEOREM: $T: \mathbb{R}^n \to \mathbb{R}^n$ LINEAR,

$\mathcal{B} = \{\underline{v}_1, \ldots, \underline{v}_n\}$ A BASIS OF $\mathbb{R}^n$

A = MATRIX OF T WITH RESPECT TO $\{\underline{e}_1, \ldots, \underline{e}_n\}$ THE STANDARD BASIS

$S = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}$, B = MATRIX OF T WITH RESPECT TO COORDINATES IN $\{\underline{v}_1, \ldots, \underline{v}_n\}$.

THEN

$$B = S^{-1}AS.$$

WE SAY THE MATRICES A, B ARE SIMILAR SINCE THEY REPRESENT THE SAME LINEAR TRANSFORMATION BUT IN DIFFERENT COORDINATES OR BASES.

PROOF:

$$A = \begin{bmatrix} | & & | \\ T\underline{e}_1 & \cdots & T\underline{e}_n \\ | & & | \end{bmatrix}. \qquad \boxed{B = S^{-1}AS.}$$

$$B = \begin{bmatrix} | & & | \\ [T\underline{v}_1]_{\mathcal{B}} & \cdots & [T\underline{v}_n]_{\mathcal{B}} \\ | & & | \end{bmatrix}.$$

$$\underline{x} = S \cdot [\underline{x}]_{\mathcal{B}}$$

$$S^{-1}\underline{x} = [\underline{x}]_{\mathcal{B}}$$

$$B = S^{-1}\begin{bmatrix} | & | & & | \\ T\underline{v}_1 & T\underline{v}_2 & \cdots & T\underline{v}_n \\ | & | & & | \end{bmatrix} = S^{-1}\begin{bmatrix} | & | & & | \\ T\underline{e}_1 & T\underline{e}_2 & \cdots & T\underline{e}_n \\ | & | & & | \end{bmatrix} \cdot \underbrace{\begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{bmatrix}}_{= S}.$$

$$T\underline{v}_i = \sum_j (\underline{v}_i)_j T\underline{e}_j$$

MULTIPLY BY S: $\mathcal{B}$ COOR TO STANDARD COOR.

B: T IN $\mathcal{B}$ COOR

IS EXPRESSED:
1. SWITCH $\mathcal{B}$ COOR TO STANDARD COOR
2. ACT BY T IN STANDARD COOR
3. SWITCH BACK TO $\mathcal{B}$ COOR.

$$B = S^{-1}AS$$

MULTIPLY BY S $\mathcal{B}$ COOR TO STANDARD COOR.

B: T IN $\mathcal{B}$ COOR

IS EXPRESSED: ① SWITCH $\mathcal{B}$-COOR TO STANDARD COOR.
② ACT BY T IN STANDARD COOR
③ SWITCH BACK TO $\mathcal{B}$ COOR.

$$B = \underset{③}{S^{-1}}\,\underset{②}{A}\,\underset{①}{S}$$

THEOREM: SIMILARITY OF MATRICES IS AN EQUIVALENCE RELATION, $\sim$ THAT IS,

a. $A \sim A$

b. $A \sim B \Rightarrow B \sim A$.

c. $A \sim B$ AND $B \sim C \Rightarrow A \sim C$.

PROOF: (a) $A = I_n^{-1} A I_n$ ✓ $A \sim A$. (b) IF $B = S^{-1}AS$, $SBS^{-1} = A$ SO $B \sim A$.

(c) $B = S^{-1}AS$

$C = T^{-1}BT$

$\Rightarrow C = T^{-1}S^{-1}AST$

$= (ST)^{-1}A(ST)$

SO $A \sim C$. $\square$

(c) $B = S^{-1}AS$

$C = T^{-1}BT$

$\Rightarrow C = T^{-1}S^{-1}AST$

$= (ST)^{-1}A(ST)$

SO $A \sim C$. $\square$

IF YOU HAVE AN EQUIVALENCE RELATION ON A SET, THE SET SPLITS INTO CLASSES

THEOREM: IF $T: \mathbb{R}^n \to \mathbb{R}^n$ IS A LINEAR MAP.

$T\underline{x} = A\underline{x}$ IN STANDARD COORDINATES.

$\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$ A BASIS.

THE MATRIX OF T W.R.T. $\mathcal{B}$ IS <u>DIAGONAL</u> $= \begin{bmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & \ddots & \\ & & & c_n \end{bmatrix}$ IF AND ONLY IF

$T\underline{v}_i = c_i\underline{v}_i$ "EIGENVECTOR WITH EIGENVALUE c_i". IN THIS CASE, A IS <u>DIAGONALIZABLE</u>

DEFINITION: A LINEAR SPACE V

IS A SET WITH A $+$, SCALAR MULTIPLICATION

SUCH THAT

(1) $(f+g)+h = f+(g+h)$.

(2) $f+g = g+f$.

(3) THERE EXISTS $\underline{0}$, $0+f = f+\underline{0}$ ALL $f \in V$.

(4) FOR EACH $f \in V$ THERE EXISTS $(-f)$ $f+(-f) = 0$

DEFINITION: A LINEAR SPACE V

IS A SET WITH A $+$, SCALAR MULTIPLICATION SUCH THAT

(1) $(f+g)+h = f+(g+h)$.

(2) $f+g = g+f$.

(3) THERE EXISTS $\underline{0}$, $0+f = f+0$ ALL $f \in V$.

(4) FOR EACH $f \in V$ THERE EXISTS $(-f)$, $f+(-f) = \underline{0}$.

$k \in \mathbb{R}$, $f, g \in V$.

(5) $k \cdot (f+g) = kf + kg$.

(6) $(c+k) \cdot f = c \cdot f + k \cdot f$ $c, k \in \mathbb{R}$, $f \in V$.

(7) $c(kf) = (ck)f$ $c, k \in \mathbb{R}$, $f \in V$.

(8) $1 \cdot f = f$.

VECTOR SPACE AXIOMS.

EXAMPLE VECTOR SPACES: THE SPACE OF ALL POLYNOMIALS

CONTINUOUS OR DIFFERENTIABLE FUNCTIONS ON $[0,1]$, 2×2 MATRICES.

EXAMPLE: $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. $\mathbb{R}^{2,2} = \{2 \times 2 \text{ REAL MATRICES}\}$

4 DIMENSIONAL $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$. $A^2 = 2I + 3 \cdot A$.

YOU CAN WRITE A^2 AS A LINEAR COMBINATION OF $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ AND A

FOR ANY MATRIX $A \in \mathbb{R}^{2,2}$, SPAN$\{I, A, A^2, A^3, A^4, A^5, \ldots\}$

EXAMPLE: $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. $\mathbb{R}^{2,2} = \{2\times 2 \text{ REAL MATRICES}\}$

4 DIMENSIONAL $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^2 = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$. $A^2 = 2I + 3 \cdot A$.

YOU CAN WRITE A^2 AS A LINEAR COMBINATION OF $\underset{I}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$ AND A.

FOR ANY MATRIX $A \in \mathbb{R}^{2,2}$, $\text{SPAN}\{I, A, A^2, A^3, A^4, A^5, \dots\}$

1 OR 2 DIMENSIONAL.

DEFINITION: A SUBSPACE OF A VECTOR SPACE V

IS A SET $W \subset V$ SATISFYING

(a) $\underline{0} \in W$

(b) IF $\underline{u}, \underline{v} \in W$, $\underline{u} + \underline{v} \in W$.

(c) IF $a \in \mathbb{R}$, $\underline{u} \in W$, THEN $a \cdot \underline{u} \in W$.

EXAMPLE: $\mathcal{P}_2$ POLYNOMIALS IN x, DEGREE ≤ 2

IS A SUBSPACE OF

- FUNCTIONS $\mathbb{R} \to \mathbb{R}$.
- C^∞ FUNCTIONS $\mathbb{R} \to \mathbb{R}$.
- POLYNOMIALS

[illegible] OF DEG ≤ 3

<u>EXAMPLE:</u> $\mathcal{P}_2$ POLYNOMIALS IN x, DEGREE ≤ 2

IS A SUBSPACE OF

- FUNCTIONS $\mathbb{R} \to \mathbb{R}$.
- C^∞ FUNCTIONS $\mathbb{R} \to \mathbb{R}$.
- POLYNOMIALS
- POLYS OF DEG ≤ 3.

<u>DEFINITION:</u> GIVEN V A LINEAR SPACE,

$f_1, \dots, f_n \in V$

(a) $f_1, \dots, f_n$ <u>SPAN</u> V IF $V = \{ c_1 f_1 + \dots + c_n f_n : c_1, \dots, c_n \in \mathbb{R} \}$

ALL LIN. COMB. OF $f_1, \dots, f_n$

(b) f_i IS <u>REDUNDANT</u> IF f_i IS A LINEAR COMBINATION OF $f_1, \dots, f_{i-1}$.

(c) $f_1, \dots, f_n$ ARE LINEARLY INDEP. IF $c_1 f_1 + \dots + c_n f_n = 0 \Rightarrow c_1 = c_2 = \dots = c_n = 0$.

(d) $f_1, \dots, f_n$ ARE A BASIS FOR V IF LINEARLY INDEPENDENT + SPAN.

$c_1, \dots, c_n$ ARE <u>COORDINATES</u> OF $f \in V$ WITH RESPECT TO BASIS $\mathcal{B} = \{ \underline{v}_1, \dots, \underline{v}_n \}$ IF

$f = c_1 f_1 + \dots + c_n f_n$. THEN $[f]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

(d) $f_1, \dots, f_n$ ARE A BASIS FOR V IF
LINEARLY INDEPENDENT + SPAN.

$c_1, \dots, c_n$ ARE <u>COORDINATES</u> OF $f \in V$ WITH RESPECT TO BASIS $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$ IF

$$f = c_1 f_1 + \dots + c_n f_n. \quad \text{THEN } [f]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

THERE IS A MAP

$$L : V \to \mathbb{R}^n$$

$$f \mapsto [f]_{\mathcal{B}}.$$